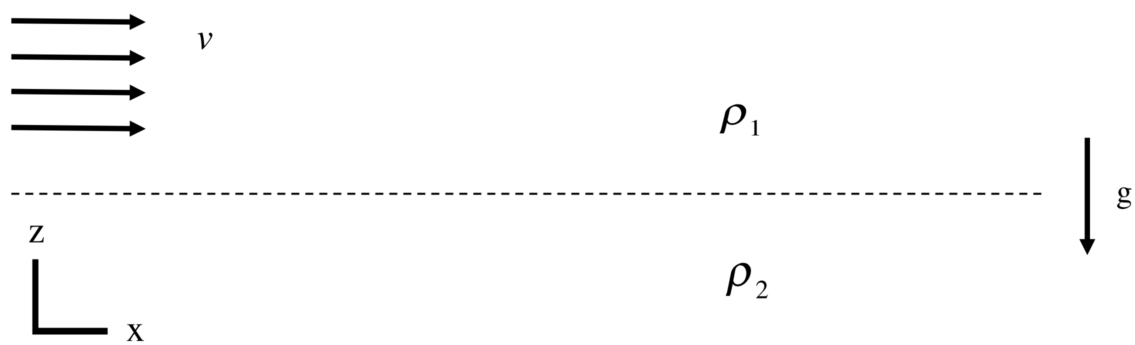
Problem Set 3: due Thursday, February 15, 2018

N.B.: These problems include pieces which are open-ended. Feel free to ask for advice, clarification! Some recommended references have been mentioned in class, in posted notes, and given in posted articles.

1) Consider a shear layer in a stably stratified fluid, as shown. Take the coefficient of surface tension between the fluids of mass densities ρ_1, ρ_2 $(\rho_1 < \rho_2)$ to be σ.



a) Calculate the general dispersion relation for waves/instabilities at the interface. Take the fluids as ideal. What controls high and low k behavior?

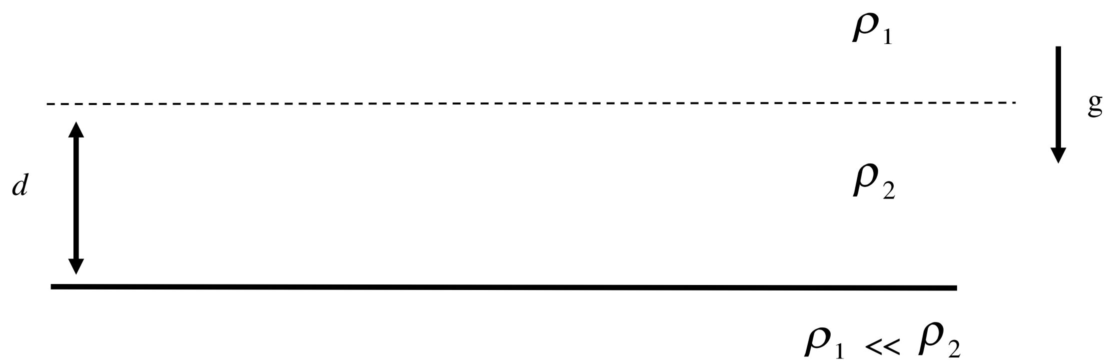
b) Ignoring surface tension, can you identify a dimensionless number which characterizes the competition between shear and buoyancy? Compare your number to the Richardson number.

c) What is the critical velocity for the onset of shear instability? Hoe does it scale with σ, ρ_1, ρ_2, k, etc.?

d) Taking $\rho_1 \leftrightarrow \text{air}$, $\rho_2 \leftrightarrow \text{water}$, this problem becomes a crude model of the air-sea interface Using it, propose a mechanism for wave generation by wind. What is the critical wind velocity for excitation of short wavelength ($\lambda \sim cm$) gravity-capillary waves?

e) The actual mechanism for wave excitation – to the extent it is understood – is nonlinear. In general terms, how might you critique your own proposal in (d.)?

2a) Determine the general dispersion relation for surface waves in a fluid of finite depth d. Treat the fluid as ideal.



b) Discuss the limits $kd >> 1$, $kd << 1$.

c) For $kd << 1$, deduce by analogy with sound waves the equations describing surface waves in shallow water. Hint: the dynamical fields are water height and horizontal velocity. Try to deduce/guess the nonlinear equations, called shallow water equations.

d) Comment on the relevance of shallow water dynamics to the objective of ripple tank demonstrations, frequently used to stimulate optical wave phenomena in high school physics classes.

3) In MHD, the Ohm's Law is

$$\underline{E}+\frac{\underline{v}\times\underline{B}}{c}=\eta\underline{J}$$

and displacement current is neglected (low frequency!), so – with Faraday's Law – one obtains the magnetic induction equation, which closely resembles the vorticity equation.

a) Derive the magnetic field induction equation. Show $\underline{B}/\rho$ is frozen-in for compressible ideal MHD.

b) For ideal MHD, prove Alfven's Theorem:

$$\oint_s \underline{B}\cdot d\underline{a}=const.$$

Be sure to treat motion of the loop. What is this the counterpart of?

c) What does Alfven's theorem mean?

4a) Derive the dispersion relation for buoyancy waves in a stably stratified fluid with $\partial S/\partial Z>0$ and $g=-g\tilde{z}$. These are called internal waves. Take the equilibrium hydrostatic. Show that internal waves are 'backward', i.e. the phase and group velocity can be in opposite directions.

b) Generalize your analysis of internal waves to include rotation effects, where $\Omega=\Omega\hat{z}$. When are corrections to the dispersion relation due to rotation of significance?

5) Falkovich observes that the interfacial version of the ideal flow shear driven instability (i.e. the Kelvin-Helmholtz instability) necessarily has a maximum (or minimum) in the profile of vorticity located at the interface. This problem addresses the presence of inflection points in smooth profiles leading to ideal shear flow instabilities.

Consider an inviscid incompressible shear flow $\underline{\mathrm{V}} = \mathrm{Vy}(x)\hat{y}$ in a domain $0 \leq \times \leq a$, $-\infty < y < +\infty$. Show that for instability to occur, there must be at least one value of x in $[0,a]$ for which $\partial^2 \mathrm{Vy}/\partial x^2 = 0$, i.e. there must be an inflection point in the flow. It is useful to approach this using the 2D vorticity advection equation and to write $\underline{\mathrm{v}} = \nabla\phi \times \hat{z}$. Also, write the frequency ω as $\omega_{\mathrm{real}} + i\gamma$.

N.B.: The theorem you just derived was first proved by Rayleigh (who else?) and establishes only that an inflection point is necessary for instability. A second theorem, due to Fjortoft, demonstrates that a vorticity *maximum* is necessary.